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COHERENT LIFE FUNCTIONS

by

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Abstract

The life function of a system expresses the life length of the system in terms of the life lengths of its components. This paper illustrates the application of life functions to reliability problems. The principal results are two characterizations of the life functions of coherent systems, which are used to obtain a number of properties for such systems.

1. Introduction.

The life length of most of the systems considered in reliability theory can be expressed as some function of component life lengths. For example if the system consists of two active components acting in parallel, then its *life (length) function* is $\tau(t_1, t_2) = \max(t_1, t_2)$, where t_1 and t_2 are the component life lengths. If one component is active and the other is a standby spare, then the system life function is $\tau(t_1, t_2) = t_1 + t_2$.

That life functions are a useful way of describing systems can be illustrated by a well known comparison between the two systems mentioned above, i.e. that the standby system is more reliable than the parallel system for any probability distribution for the component life lengths simply because $\max(t_1, t_2) \leq t_1 + t_2$ for all $t_1, t_2 \geq 0$.

The parallel system is a simple representative of the class of coherent systems [a survey of coherent systems is given by Barlow and Proschan, 1965]. The standby system cannot be considered a coherent system with two genuine components (Remark 2.8). This paper is devoted to a discussion of the life functions of coherent systems. The principal results are two useful characterizations of coherent life functions (Theorem 2.6). Sample applications of the characterizations are given (Section 5). Since coherent systems can be defined in terms of their life functions, some basic auxiliary definitions are similarly stated (Section 4). Since a number of the properties of coherent life functions hold for the life functions of other systems, some of these properties are examined (Section 3).

2. Characterizations.

We assume that all the components in a system have life lengths $t_1 \geq 0, \dots, t_n \geq 0$ in the sense that the i^{th} component is functioning at all times $u < t_i$ and failed at all times $u \geq t_i, u \geq 0$. With

$$(2.1) \quad x(u, t) = \begin{cases} 1 & \text{if } u < t \\ 0 & \text{if } u \geq t, \end{cases}$$

and $x(u, \underline{t}) = \{x(u, t_1), \dots, x(u, t_n)\}$, we also assume that there is a binary function ϕ of binary arguments $x = \{x_1, \dots, x_n\}$ for which $\phi\{x(u, \underline{t})\} = 1$ if the system is functioning at time u and $\phi\{x(u, \underline{t})\} = 0$ if the system is failed at time u . ϕ is a *structure function*. The system has a *coherent* structure if ϕ is increasing, i.e. $\phi\{\underline{x}\} \leq \phi\{\underline{y}\}$ whenever $x_i \leq y_i, i = 1, \dots, n, \phi\{\underline{1}\} = 1$ where $\underline{1} = \{1, \dots, 1\}$, and $\phi\{\underline{0}\} = 0$ where $\underline{0} = \{0, \dots, 0\}$.

For a system with a coherent structure, $\phi\{x(u, \underline{t})\}$ is decreasing in u for each $\underline{t} = \{t_1, \dots, t_n\}$, since $x(u, t)$ is decreasing in u for each t . Thus the system has a life length $\tau\{\underline{t}\}$ given by

$$(2.2) \quad \tau\{\underline{t}\} = \sup \{u : \phi\{x(u, \underline{t})\} = 1\}.$$

Since $x(u, t)$ is continuous from the right in u for each t , $\phi\{x(u, \underline{t})\}$ is continuous from the right in u for each \underline{t} . Thus (2.2) is equivalent to

$$(2.3) \quad \tau\{\underline{t}\} > u \Leftrightarrow \phi\{x(u, \underline{t})\} = 1$$

holding for all $u \geq 0$. The function $\tau\{\underline{t}\}, t_i \geq 0, i = 1, \dots, n$,

is the *life (length) function* of the system. τ is an extension of ϕ , i.e.

$$(2.4) \quad \tau(\underline{x}) = \phi(\underline{x}) \quad \text{for all } \underline{x},$$

where $x_i = 0$ or 1 , $i = 1, \dots, n$.

The following lemma will be useful in obtaining two fundamental characterizations of the life functions of systems with a coherent structure.

2.5 Lemma. Let $\tau(\underline{t})$ be a function defined for $t_i = 0, a_i, 1$ where $0 < a_i < 1$, $i = 1, \dots, n$. With $\xi(\underline{t}) = \{\xi(t_1), \dots, \xi(t_n)\}$, suppose that $\xi(\tau(\underline{t})) = \tau(\xi(\underline{t}))$ for all values of \underline{t} and all binary, increasing functions ξ . Then $\tau(\underline{x})$, the restriction of τ to binary arguments x_i , $i = 1, \dots, n$, is a coherent structure function.

Proof. Let ξ be a binary, increasing function such that $\xi(0) = 0$ and $\xi(1) = 1$. Then for all \underline{x} , $\tau(\underline{x}) = \tau(\xi(\underline{x})) = \xi(\tau(\underline{x})) = 0$ or 1 , i.e. τ is a binary function. Let $\xi(u) = 0$ for all u . Then for any \underline{t} , $\tau(\underline{0}) = \tau(\xi(\underline{t})) = \xi(\tau(\underline{t})) = 0$. The same argument, with the function $\xi \equiv 1$, shows that $\tau(\underline{1}) = 1$. It remains to show that $\tau(\underline{x})$ is increasing, i.e. that $\tau(\underline{0}_i, \underline{x}) \leq \tau(\underline{1}_i, \underline{x})$ for all i and \underline{x} , where $\{c_i, \underline{x}\} = \{x_1, \dots, x_{i-1}, c, x_{i+1}, \dots, x_n\}$. Let ξ and η be binary, increasing functions such that $\xi(0) = \xi(a_i) = 0$, $\xi(1) = 1$ and $\eta(0) = 0$, $\eta(a_i) = \eta(1) = 1$. Note that $\xi(u) \leq \eta(u)$ for all u . Then $\tau(\underline{0}_i, \underline{x}) = \tau(\xi(a_i), \xi(\underline{x})) = \xi(\tau(\underline{a}_i, \underline{x})) \leq \eta(\tau(\underline{a}_i, \underline{x})) = \tau(\eta(a_i), \eta(\underline{x})) = \tau(\underline{1}_i, \underline{x})$. Thus $\tau(\underline{x})$ is increasing. \square

2.6 Theorem. Let $\tau(\underline{t})$ be defined for $t_i \geq 0, i = 1, \dots, n$. Then the following three conditions are equivalent:

- (i) τ is the life function of a system with a coherent structure.
- (ii) For each simplex $0 \leq t_{i_1} \leq t_{i_2} \leq \dots \leq t_{i_n}$ there is an i such that $\tau(\underline{t}) = t_i$ everywhere on the simplex.
- (iii) $f(\tau(\underline{t})) = \tau(f(\underline{t}))$, where $f(\underline{t}) = \{f(t_1), \dots, f(t_n)\}$, for all \underline{t} and all positive ($f \geq 0$), increasing functions f .

Proof. (i) \Rightarrow (ii): It is easy to see that $\phi\{\underline{x}\} = x_i$ for some i everywhere that $x_{i_1} \leq x_{i_2} \leq \dots \leq x_{i_n}$. On a simplex $0 \leq t_{i_1} \leq t_{i_2} \leq \dots \leq t_{i_n}$, $x(u, t_{i_1}) \leq x(u, t_{i_2}) \leq \dots \leq x(u, t_{i_n})$ for all u , since from (2.1) $x(u, t)$ is increasing in t for each u . Then $\phi\{x(u, \underline{t})\} = x(u, t_i)$ for some i and all u and all \underline{t} in the simplex. Then by (2.3) and (2.1), $\tau(\underline{t}) > u \Leftrightarrow x(u, t_i) = 1 \Leftrightarrow t_i > u$ for all u , i.e. $\tau(\underline{t}) = t_i$ everywhere on the simplex.

(ii) \Rightarrow (iii): For each $\underline{t}, \underline{t'}$ and the vector $f(\underline{t})$ are in the same simplex since f is increasing. Since $\tau(\underline{t}) = t_i$ for some i and $\tau\{f(\underline{t})\} = f(t_i)$ for the same i , then $f(\tau(\underline{t})) = f(t_i) = \tau\{f(\underline{t})\}$.

(iii) \Rightarrow (i): $\tau\{\underline{x}\}$ is a coherent structure function by Lemma 2.5. Since $x(u, t)$ is increasing in t for each u , $x(u, \tau(\underline{t})) = \tau\{x(u, \underline{t})\}$ for all u and all \underline{t} . Then from (2.1), $\tau(\underline{t}) > u \Leftrightarrow \tau\{x(u, \underline{t})\} = 1$ for all u . Thus from (2.3), τ is the life function of a system with a coherent structure. \square

The following remark shows that in general a coherent life function is equal to its i^{th} coordinate on the union of several adjacent simplexes.

2.7 Remark. Let τ be the life function of a system with a coherent structure. If $\tau\{\underline{s}\} = s_i$ for some \underline{s} such that $s_j \neq s_i$ for all $j \neq i$, then $\tau\{\underline{t}\} = t_i$ for all \underline{t} such that $t_j \leq t_i$ if $s_j \leq s_i$ and $t_j \geq t_i$ if $s_j \geq s_i$.

Proof. Choose any a, b such that $a < s_i < b$. Let $u_i = s_i$ and $u_j = a$ if $s_j < s_i$, $u_j = b$ if $s_j > s_i$, $j \neq i$. Then \underline{u} and \underline{s} are in the same simplex. Since $\tau\{\underline{s}\} = s_i$ and $\tau\{\underline{s}\} \neq s_j$, $j \neq i$, then $\tau\{\underline{u}\} = u_i$ by part (ii) of Theorem 2.6. \underline{u} is in the simplex of any \underline{t} described above. Again since $\tau\{\underline{u}\} = u_i$ and $\tau\{\underline{u}\} \neq u_j$, $j \neq i$, then $\tau\{\underline{t}\} = t_i$. \square

In terms of its structure function ϕ , the i^{th} component of a coherent system is *essential* if $\phi\{1_i, \underline{x}\} = 1$ and $\phi\{0_i, \underline{x}\} = 0$ for some vector \underline{x} , where $(c_i, \underline{x}) = (x_1, \dots, x_{i-1}, c, x_{i+1}, \dots, x_n)$. A system has a *fully coherent* structure if it has a coherent structure and each component is essential.

2.8 Remark. Let τ be the life function of a system with a coherent structure. The i^{th} component is essential if and only if $\tau\{\underline{t}\} = t_i$ for some \underline{t} such that $t_j \neq t_i$ for all $j \neq i$.

Proof. (only if). Consider any \underline{t} such that $t_j < t_i$ if $x_j = 0$, $t_j > t_i$ if $x_j = 1$, $j \neq i$. Let $\xi(u) = 1$ if $u = t_i$, $\xi(u) = 0$ if

$u \leq t_i$. Let $\eta(u) = 1$ if $u \geq t_i$, $\eta(u) = 0$ if $u < t_i$. Then $\xi(\underline{t}) = \{0_i, \underline{x}\}$ and $\eta(\underline{t}) = \{1_i, \underline{x}\}$. From part (iii) of Theorem 2.6 and (2.4), $\xi(\tau(\underline{t})) = \tau(\xi(\underline{t})) = \phi\{0_i, \underline{x}\} = 0$. Similarly $\eta(\tau(\underline{t})) = \phi\{1_i, \underline{x}\} = 1$. Since $\xi(u) \neq \eta(u)$ only when $u = t_i$, then $\tau(\underline{t}) = t_i$.

(if). Let $x_j = 0$ if $t_j < t_i$, $x_j = 1$ if $t_j = t_i$. Let ξ and η be defined as above. Then $\phi\{0_i, \underline{x}\} = \xi(\tau(\underline{t})) = \xi(t_i) = 0$ and $\phi\{1_i, \underline{x}\} = \eta(t_i) = 1$. \square

3. Some properties.

The life function τ of a system with a coherent structure has a number of elementary properties that are descriptive of the behavior of such systems. A summary of some of these properties is given here:

$$(3.1) \quad \tau \text{ is increasing, i.e. } \tau\{\underline{s}\} \leq \tau\{\underline{t}\} \text{ whenever } s_i \leq t_i, \\ i = 1, \dots, n.$$

Since $x(u, t)$ is increasing in t for each u and ϕ is increasing, (3.1) follows from (2.2).

$$(3.2) \quad \tau \text{ is homogeneous, i.e. } \tau\{\alpha \underline{t}\} = \alpha \tau\{\underline{t}\} \text{ for all } \underline{t} \text{ and} \\ \text{all } \alpha \geq 0, \text{ where } \alpha \underline{t} = \{\alpha t_1, \dots, \alpha t_n\}.$$

With $f(u) = \alpha u$, (3.2) follows from part (iii) of Theorem 2.6. (3.2) can also be obtained from part (ii) of the same theorem.

$$(3.3) \quad \tau\{\underline{t} + \delta \underline{1}\} = \tau\{\underline{t}\} + \delta \text{ for all } \underline{t} \text{ and all } \delta \geq 0, \text{ where} \\ \underline{t} + \delta \underline{1} = \{t_1 + \delta, \dots, t_n + \delta\}.$$

With $f(u) = u + \delta$, (3.3) follows from part (iii) of Theorem 2.6. (3.3) can be obtained from part (ii) of the same theorem.

$$(3.4) \quad \tau\{\delta \cdot \underline{1}\} = \delta \text{ for all } \delta \geq 0, \text{ and in particular } \tau\{\underline{0}\} = 0.$$

With $\underline{t} = \underline{0}$, (3.4) follows from (3.3). (3.4) is immediate from part (ii) of Theorem 2.6. That $\tau\{\underline{0}\} = 0$ also follows from (2.4).

$$(3.5) \quad \tau \text{ is positive, i.e. } \tau\{\underline{t}\} \geq 0 \text{ for all } \underline{t}.$$

Since $\tau\{\underline{0}\} = 0$, (3.5) follows from (3.1). (3.5) is immediate from part (ii) of Theorem 2.6.

$$(3.6) \quad \min_{i=1, \dots, n} s_i \leq \tau(\underline{t} + \underline{s}) - \tau(\underline{t}) \leq \max_{i=1, \dots, n} s_i$$

for all \underline{s} and \underline{t} , where $\underline{t} + \underline{s} = \{t_1 + s_1, \dots, t_n + s_n\}$.

Using (3.1), (3.6) follows from (3.3) with $\delta = \min_{i=1, \dots, n} s_i$ and $\delta = \max_{i=1, \dots, n} s_i$. (3.6) implies (3.3).

$$(3.7) \quad \tau \text{ is continuous.}$$

The continuity of τ is immediate from (3.6).

4. Path and cut representation.

The life function τ of a system with a coherent structure function ϕ can be conveniently represented in terms of either the minimal paths or the minimal cuts of the system structure. A *minimal path set* is a minimal set P such that $\min_{i \in P} x_i = 1$ implies $\phi(\underline{x}) = 1$, or equivalently a minimal set P such that $\min_{i \in P} t_i > u$ implies $\tau(\underline{t}) > u$ for all $u \geq 0$. A *minimal cut set* is a minimal set K such that $\max_{i \in K} x_i = 0$ implies $\phi(\underline{x}) = 0$, or equivalently a minimal set K such that $\max_{i \in K} t_i \leq u$ implies $\tau(\underline{t}) \leq u$ for all $u \geq 0$. It is easy to see that a coherent structure function can be represented by

$$(4.1) \quad \begin{aligned} \phi(\underline{x}) &= \min_{j=1, \dots, p} \max_{i \in P_j} x_i \\ &= \max_{j=1, \dots, k} \min_{i \in K_j} x_i, \end{aligned}$$

where P_1, \dots, P_p are the minimal path sets and K_1, \dots, K_k are the minimal cut sets [Birnbaum, Esary, and Saunders, 1961]. The corresponding representation for τ is

$$(4.2) \quad \begin{aligned} \tau(\underline{t}) &= \min_{j=1, \dots, p} \max_{i \in P_j} t_i \\ &= \max_{j=1, \dots, k} \min_{i \in K_j} t_i. \end{aligned}$$

Edmonds and Fulkerson (1968), working with bottleneck problems, show that (4.2) characterizes an equivalent of systems with a coherent structure. To confirm their result note that any τ represented by (4.2) satisfies the characterizations of parts (ii) and (iii) of Theorem 2.6.

5. Sample reliability applications.

The characterizations of the life function τ of a system with a coherent structure given in Theorem 2.6 are useful in proving certain results about the stochastic behavior of such systems. Two examples are mentioned here.

Let \mathcal{A} be a class of probability distributions for positive ($T \geq 0$) random variables T (life distributions), and let the notation $T \in \mathcal{A}$ mean that the distribution of T is in \mathcal{A} . \mathcal{A} is *closed under the formation of coherent systems* if $\tau(\underline{T}) \in \mathcal{A}$, where $\underline{T} = \{T_1, \dots, T_n\}$, for all coherent life functions τ whenever T_1, \dots, T_n are statistically independent and $T_i \in \mathcal{A}$, $i = 1, \dots, n$. This closure operation has been discussed by Birnbaum, Esary, and Marshall (1966) in connection with the increasing hazard rate average (closed) class of life distributions. For a measurable function f let $f(\mathcal{A})$ denote the class of all probability distributions for the random variables $f(T)$, $T \in \mathcal{A}$.

5.1 Application. Let \mathcal{A} be closed under the formation of coherent systems, and f be a positive, increasing function. Then $f(\mathcal{A})$ is closed under the formation of coherent systems.

Proof. It is necessary to show that $\tau(\underline{T}) \in f(\mathcal{A})$ for all coherent τ whenever T_1, \dots, T_n are independent and $T_i \in f(\mathcal{A})$, $i = 1, \dots, n$. Let $T_i = f(U_i)$, $i = 1, \dots, n$, where U_1, \dots, U_n are independent and $U_i \in \mathcal{A}$. From part (iii) of Theorem 2.6, $\tau(\underline{T}) = \tau\{f(\underline{U})\} = f(\tau(\underline{U}))$.

Since \mathcal{A} is closed, $\tau\{\underline{U}\} \in \mathcal{A}$. Thus $f(\tau\{\underline{U}\}) \in f(\mathcal{A})$, i.e.

$\tau\{\underline{T}\} \in f(\mathcal{A})$. \square

It follows from part (ii) of Theorem 2.6 that for any set A , if $t_i \in A$, $i = 1, \dots, n$, then $\tau\{\underline{t}\} \in A$, and if $\tau\{\underline{t}\} \in A$, then $t_i \in A$ for some i . Thus if A is measurable and $T_1 \geq 0, \dots, T_n \geq 0$ are random variables, then

$$(5.2) \quad P\left[\bigcap_{i=1}^n [T_i \in A]\right] \leq P[\tau\{\underline{T}\} \in A] \leq P\left[\bigcup_{i=1}^n [T_i \in A]\right].$$

5.3 Application. Let T_1, \dots, T_n be absolutely continuous, positive random variables. Let τ be the life function of a system with a coherent structure. Then $\tau\{\underline{T}\}$ is absolutely continuous.

Proof. Let A be a set whose Lebesgue measure is zero. Since T_i is absolutely continuous, $P[T_i \in A] = 0$, $i = 1, \dots, n$. Then from (5.2)

$$P[\tau\{\underline{T}\} \in A] \leq P\left[\bigcup_{i=1}^n [T_i \in A]\right] \leq \sum_{i=1}^n P[T_i \in A] = 0,$$

i.e. $\tau\{\underline{T}\}$ is absolutely continuous. \square

Application 5.3 does not require that T_1, \dots, T_n be independent, or that they be jointly absolutely continuous.

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